CONVERGENCE CRITERIA FOR DOUBLE FOURIER SERIES*

BY

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1.1. **Introduction.** We shall consider here analogues for double Fourier series of certain convergence criteria for simple Fourier series. The tests for simple series in question are the familiar tests of Dini, Jordan, de la Vallée-Poussin, Lebesgue, Young, and Hardy and Littlewood, and the tests obtained by various authors in generalizing the Young and the Lebesgue test. All these criteria are stated, the logical relations between them discussed, and references to them given in the author's paper **4**.‡ Rather than duplicate this material here we refer the reader to that paper. Analogues for some of these tests have appeared in the literature. Our first purpose here is to establish analogues of those remaining. Our second purpose is to discuss the logical relations between the tests for double series. We obtain, incidentally, an extension of Tonelli's convergence criterion for double series which deals with functions of bounded variation. Statements of our results and a general summary of the convergence theory are to found in §§1.2 to 1.6, the proofs of our theorems, in §§2.0 to 13.1. We do not always attempt to model the proof of a generalization after the proof of the original; but deduce first a test of the Lebesgue type, and from it the other tests. We thus obtain at the same time information as to the relations between the tests.

1.2. We suppose once and for all that the double Fourier series in question is that of an even-even function $f(u, v)$ which is integrable in the Lebesgue sense over the square $Q(0, 0; \pi, \pi)$ and is doubly periodic with period 2π in each variable. Further, we shall confine our attention to the behavior of the Fourier series of f at the origin. We have, then,

$$f(u, v) \sim \sum_{m,n=0}^{\infty} \lambda_{m,n} a_{m,n} \cos mu \cos nv,$$

where

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‡ Numbers in bold face type refer to the Bibliography at the end of this paper.

$$\lambda_{0,0} = \tfrac{1}{4}, \quad \text{and } \lambda_{m,0} = \lambda_{0,n} = \tfrac{1}{2}, \quad \lambda_{m,n} = 1 \text{ for } 0 < m,\ 0 < n,$$

and

$$a_{m,n} = \frac{4}{\pi^2}\int\int_Q f(u, v) \cos mu \cos nv\, dudv.$$

The series for f at the origin is

$$\sum_{m,n=0}^{\infty} \lambda_{m,n} a_{m,n}; \tag{1.21}$$

the partial sum $s_{m,n}$ of order m, n of this series is

$$s_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} \lambda_{i,j} a_{i,j} = \frac{1}{\pi^2}\int\int_Q f(u, v)\frac{\sin (m+\frac{1}{2})u}{\sin \frac{1}{2}u}\ \frac{\sin (n+\frac{1}{2})v}{\sin \frac{1}{2}v}\, dudv;$$

and we are concerned with the limit

$$\lim_{m,n\to\infty} s_{m,n}$$

taken in the Pringsheim sense.* Any test for the convergence of the series (1.21) yields, of course, a test for convergence of the Fourier series of an arbitrary integrable function at an arbitrary point.

1.3. To simplify the writing we employ a form of the Landau limit notation. Given two functions $h(x, y)$ and $\psi(x, y)$, defined for all sufficiently small positive values of x and y, we write

$$h(x, y) = o\{\psi(x, y)\} \tag{1.31}$$

if, corresponding to each number $0<\epsilon$, we can choose $0<\delta_\epsilon$ so that

$$|h| \leqq \epsilon|\psi| \tag{1.32}$$

for $0<x\leqq\delta_\epsilon$, $0<y\leqq\delta_\epsilon$. We write

$$h(x, y) = O\{\psi(x, y)\} \tag{1.33}$$

if (1.32) holds for some ϵ and all sufficiently small positive values of x and y. Given two functions $h(x, y; k)$ and $\psi(x, y; k)$, defined for each large value of k for sufficiently small positive values of x and y, we write

$$h(x, y; k) = \bar{o}\{\psi(x, y; k)\} \tag{1.34}$$

* Pringsheim, 12, p. 103. The series (1.21) converges, to sum s, or

$$\lim_{m,n\to\infty} s_{m,n} = s$$

in the Pringsheim sense, if there corresponds to every number $0<\epsilon$ an integer N such that, if $N\leqq m$, $N\leqq n$, then $|s_{m,n}-s|\leqq\epsilon$.

if, corresponding to each $0<\epsilon$, we can choose, first, $0<k_\epsilon$; and then, $0<\delta_{k,\epsilon}$, so that (1.32) holds for

$$0 < x \leqq \delta_{k,\epsilon}, 0 < y \leqq \delta_{k,\epsilon}, k_\epsilon \leqq k. \tag{1.35}$$

We write

$$h(x, y; k) = \bar{O}\{\psi(x, y; k)\} \tag{1.36}$$

if, corresponding to some ϵ, we can choose k_ϵ and $\delta_{k,\epsilon}$ as above so that (1.32) holds for all x, y, and k satisfying (1.35).†

1.4. The known tests for the convergence of the series (1.21) which are of interest here may now be recalled. In stating these, and in what follows, we understand that letters capped by bars, $(\bar{D})$, $(\bar{J})$, etc., have the same meanings as the same letters without the bars in the author's paper 4. Letters without bars refer to conditions and tests for double series. In some of the tests there are two or three conditions. We shall always regard the set of conditions in any test as a single condition and denote it by the same notation as we use to denote the test itself. Similarly, when a test involves but one condition we denote it in the same way as we do the test itself.

The conditions sufficient for the convergence of the series (1.21) *are in*
(D_Y) *Young's analogue of Dini's test* $(\bar{D})$:‡

$$\int_0^\pi \frac{du}{u}\int_0^\pi |f(u, v) - s - \xi_1(u) - \xi_2(v)| \frac{dv}{v} < \infty$$

where s is a constant and ξ_1, ξ_2 are functions such that $\xi_1(u)/u$, $\xi_2(v)/v$ are integrable over $(0, \pi)$;

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† In case ψ is a non-vanishing function, then (1.31), (1.33), (1.34), (1.36) respectively holds if, and only if,

$$\lim_{(x,y)\to(+0,+0)} (h/\psi) = 0, \qquad \overline{\lim_{(x,y)\to(+0,+0)}} |h/\psi| < \infty,$$

$$\lim_{k\to\infty} \overline{\lim_{(x,y)\to(+0,+0)}} |h/\psi| = 0, \quad \overline{\lim_{k\to\infty}} \overline{\lim_{(x,y)\to(+0,+0)}} |h/\psi| < \infty.$$

‡ Young, 15, p. 182. About the same time as Young's paper appeared Küstermann, 10, p. 28, published an analogue of Dini's test. Later a test of this type was given by Merriman, 3, p. 129. The test (D_Y) is not exactly the test of any of these authors. It includes them all as particular cases however.

Young's condition is
(D_Y^*) *the function* $(f-s)/(uv)$ *is integrable over* Q.
Young proves that, if (D_Y^*) is satisfied, then so also is (V_Y^*) (see the footnote on (V_Y) above). Using Young's method it is not difficult to show that (D_Y) implies (V_Y^*). Thus (D_Y) is a sufficient condition for convergence.

(J_H) *Hardy's analogue of Jordan's test* $(\bar{J})$:†
(J_H') *f is finitely defined everywhere in Q and*

$$\int_0^\pi \int_0^\pi | d_{u,v} f(u, v) | < \infty,$$

where the integral represents the total variation of f over Q,‡ and

$$(J_H'') \qquad \int_0^\pi | d_u f(u, 0) | < \infty, \quad \int_0^\pi | d_v f(0, v) | < \infty,$$

where the first integral is the total variation of $f(u, 0)$, *and the second, the total variation of* $f(0, v)$, *over* $(0, \pi)$;§
(J_T) *Tonelli's analogue of* $(\bar{J})$:‖
(J_T') *f is finitely defined everywhere in Q and*

$$V_1(v) \equiv \int_0^\pi | d_u f(u, v) | < V(v), \quad V_2(u) \equiv \int_0^\pi | d_v f(u, v) | < V(u),$$

where V is integrable over $(0, \pi)$, *the first for every V, and the second for every u, on* $(0, \pi)$,¶ *and*

$$(J_T'') \qquad \begin{aligned} W_1(x, y) &\equiv \lim_{\tau \to +0} \int_\tau^x | d_u f(u, y) | = o(1), \\ W_2(x, y) &\equiv \lim_{\tau \to +0} \int_\tau^y | d_v f(x, v) | = o(1).** \end{aligned}$$

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† Hardy, **5**, p. 65.

‡ The total variation of f over the rectangle $(a_1, b_1; a_2, b_2)$,

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} | d_{u,v} f(u, v) |,$$

is defined as the upper bound of all sums of the type

$$\sum_{i=1}^{m} \sum_{j=1}^{n} | f(u_i, v_j) - f(u_i, v_{j-1}) - f(u_{i-1}, v_j) + f(u_{i-1}, v_{j-1}) |$$

where u_i, $i=0, 1, \cdots, m$, and v_j, $j=0, 1, \cdots, n$, are any numbers such that $a_1 = u_0 < u_1 < \cdots < u_m = a_2$, $b_1 = v_0 < v_1 < \cdots < v_n = b_2$.

§ Hardy states this second condition as

$(J_H''*)$ $\int_0^\pi | d_u f(u, v) | < \infty$, $\int_0^\pi | d_v f(u, v) | < \infty$,

the first for every v, the second for every u, on $(0, \pi)$.

It is pointed out by Young, **15**, p. 142, that, if (J_H') and (J_H'') hold, then $(J_H''*)$ holds.

‖ Tonelli, **13**, p. 455, and **14**.

¶ This condition is stated by Tonelli as

$(J_T'*)$ V_1, V_2 *are integrable over* $(0, \pi)$.

Since, as Professor Adams pointed out to the author, there are functions for which V_1 and V_2 are not measurable, there is some gain in generality in taking the condition as we do. That (J_T') and (J_T'') are sufficient conditions for convergence, we prove by Theorems I, II and III below.

When a function satisfies (J_H) it may be said to be of bounded variation in the Hardy sense, and when it satisfies (J_T'), of bounded variation in the Tonelli sense. Other definitions of bounded variation have been given by various authors. For a complete discussion of these, see Adams, **1**. For a convergence theorem involving another definition, see Hobson, **2**, p. 705. For further references on convergence criteria, see Tonelli, **14**.

** Tonelli states this condition in a slightly different but equivalent way.

(V_Y) *Young's analogue of de la Vallée-Poussin's test* $(\overline{Y})$:† *the mean value*

$$F(u, v) = \frac{1}{uv}\int_0^u d\sigma \int_0^v f(\sigma, t)dt \tag{1.41}$$

of f satisfies (J_H).

In (D_Y) *the sum of the series is s, in* (J_H) *and* (J_T), $f(+0, +0)$, *and in* (V_Y), $F(+0, +0)$.

Each of the above tests is plainly analogous to the corresponding test for simple series. There is, however, one aspect in which these tests and, in fact, all the tests given here for double series, differ from the original tests. In each test for double series there is some condition on f, other than integrability, over the whole square Q, whereas for simple series, the only conditions imposed, other than integrability, are neighborhood conditions. Now, by the analogue of the Riemann-Lebesgue theorem,‡ the behavior of f in the square $(\delta, \delta; \pi, \pi)$, provided $0<\delta$, has no effect on the convergence of the series (1.21). Thus this difference could be partially eliminated; but we cannot, as might be expected, confine our conditions to neighborhood conditions. Conditions on f in the "cross-neighborhood" of the origin are essential. Some of the above tests were originally stated with only cross-neighborhood conditions, and we could state those which follow thus. We state the tests as we do for simplicity.

From each of the above tests the corresponding test for simple series can readily be deduced. We have, when f is a function of u alone, $f=\bar{f}(u)$, say,

$$s_{m,n} = s_{m,0} = \frac{1}{\pi}\int_0^\pi \bar{f}(u)\,\frac{\sin (m+\frac{1}{2})u}{\sin \frac{1}{2}u}\,du,$$

which is the mth partial sum of the simple Fourier series of $\bar{f}$ corresponding to the point $u=0$. Now, if $\bar{f}$ satisfies $(\overline{D})$, f satisfies (D_Y); and if $\bar{f}$ satisfies $(\overline{V})$, f satisfies (V_Y). Hence from (D_Y) we can immediately deduce $(\overline{D})$, and from (V_Y), $(\overline{V})$. The passage from (J_H) and (J_T) to $(\overline{J})$ is not so immediate, but a simple application of the Riemann-Lebesgue theorem leads directly to the desired conclusion.

1.5. An examination of §1.4 reveals that the types of tests for simple series which have not been considered for double series are Young's, Hardy

† Young, 15, p. 170. Young states his condition in another but equivalent form, namely: $(V_Y{}^*)$ $Fuv \csc u \csc v$ *satisfies* (J_H).

The right-hand member of (1.41) has, of course, no meaning when $u=0$ or when $v=0$. It is implied that F can be so defined on the axes as to satisfy (J_H).

‡ For this analogue, see Young, 15, p. 138.

and Littlewood's, and Lebesgue's. Listed below is our extension of Tonelli's criterion and the analogues we obtain of tests of these types.

The conditions sufficient for the convergence of the series (1.21), *to sum s, are, in*

(J_R) *our extension of Tonelli's test* (J_T): (J_T'),

$$(J_R'') \qquad W_1(x, y) = O(1), \; W_2(x, y) = O(1),$$

and

$$(C_1) \qquad \phi_1(x, y) \equiv \int_0^x du \int_0^y \phi(u, v)dv = o(xy),$$

where $\phi = f - s$;

(Y) *our analogue of Young's test* $(\overline{Y})$:

(Y') *f is finitely defined everywhere in Q and*

$$(1.51) \quad \int_0^x \int_0^y \left| d_{u,v}\{uvf(u, v)\} \right| < Axy \text{ for } 0 < x \leqq \pi, 0 < y \leqq \pi,$$

where A is independent of x and y, and

$$(C_0) \qquad \phi(x, y) = o(1);$$

(Y_P) *our analogue of Pollard's generalization* $(\overline{Y}_{\bar{P}})$ *of* $(\overline{Y})$: (Y') *and* (C_1);

(HL) *our analogue of Hardy and Littlewood's test* $(\overline{HL})$:

$$(HL') \qquad \int_0^{\pi-x} du \int_0^{\pi-y} \left| \Delta_{x,y} f(u, v) \right|^{p_1} dv = O(xy),$$

where

$$\Delta_{x,y} f(u, v) = f(u+x, v+y) - f(u+x, v) - f(u, v+y) + f(u, v),$$

for some $1 \leqq p_1$,

$$(HL'') \quad \int_0^x du \int_0^{\pi-y} \left| \Delta_y f \right|^{p_2} dv = O(xy), \quad \int_0^y dv \int_0^{\pi-x} \left| \Delta_x f \right|^{p_3} du = O(xy),$$

where†

$$\Delta_y f = f(u, v+y) - f(u, v), \quad \Delta_x f = f(u+x, v) - f(u, v),$$

for some $1 \leqq p_2$ *and some* $1 \leqq p_3$, *and* (C_1);

† There is some confusion in the notation $\Delta_x f$, $\Delta_y f$, and $\Delta_{x,y} f$, but this is not serious. Whenever we have $\Delta_x h$, h has u as one of its arguments, and $\Delta_x h$ is the first difference

$$\Delta_x h = h(u+x \cdots) - h(u \cdots).$$

Similarly, y always appears with and is coupled with v. Finally, whenever we have $\Delta_{x,y} h$, h has u and v as two of its arguments and $\Delta_{x,y} h$ is the first double difference

$$\Delta_{x,y} h = h(u+x \cdots v+y \cdots) - h(u+x \cdots v \cdots) - h(u \cdots v+y \cdots) + h(u \cdots v \cdots).$$

(L_1) *our analogue of Lebesgue's test* $(\bar{L}_1)$:

$$(L_1') \qquad \int_x^{\pi-x} \frac{du}{u} \int_y^{\pi-y} |\Delta_{x,y} f| \frac{dv}{v} = o(1),$$

$$(L_1'') \qquad \int_0^x du \int_y^{\pi-y} |\Delta_y f| \frac{dv}{v} = o(x), \quad \int_0^y dv \int_x^{\pi-x} |\Delta_x f| \frac{du}{u} = o(y),$$

and

$$(C_1^*) \qquad \phi_1^*(x, y) \equiv \int_0^x du \int_0^y |\phi(u, v)| \, dv = o(xy);$$

(L_2) *our analogue of Lesbegue's test* $(\bar{L}_2)$:

$$(L_2') \qquad \zeta_1(x, y) \equiv \int_x^{\pi-x} du \int_y^{\pi-y} \left| \Delta_{x,y} \left\{ \frac{\phi(u, v)}{uv} \right\} \right| dv = o(1),$$

and

$$(L_2'') \qquad \begin{aligned} \xi_1 &\equiv \int_0^x du \int_y^{\pi-y} \left| \Delta_y \left\{ \frac{\phi}{v} \right\} \right| dv = o(x), \\ \eta_1 &\equiv \int_0^y dv \int_x^{\pi-x} \left| \Delta_x \left\{ \frac{\phi}{u} \right\} \right| du = o(y); \end{aligned}$$

(L_P) *our analogue of Pollard's generalization* $(\bar{L}_P)$ *of* $(\bar{L}_2)$:

$$(L_P') \qquad \zeta(x, y; k) \equiv \int_{kx}^{\pi-x} du \int_{ky}^{\pi-y} \left| \Delta_{x,y} \left\{ \frac{\phi}{uv} \right\} \right| dv = \bar{o}(1),$$

$$(L_P'') \qquad \begin{aligned} \xi &\equiv \int_0^x du \int_{ky}^{\pi-y} \left| \Delta_y \left\{ \frac{\phi}{v} \right\} \right| dv = \bar{o}(x), \\ \eta &\equiv \int_0^y dv \int_{kx}^{\pi-x} \left| \Delta_x \left\{ \frac{\phi}{u} \right\} \right| du = \bar{o}(y), \end{aligned}$$

and (C_1);

(L_R) *our analogue of Gergen's generalization* $(\bar{L}_R)$ *of* $(\bar{L}_1)$:

$$(L_R') \qquad \gamma(x, y; k) \equiv \int_{kx}^{\pi-x} \frac{du}{u} \int_{ky}^{\pi-y} |\Delta_{x,y} f| \frac{dv}{v} = \bar{o}(1),$$

$$(L_R'') \qquad \begin{aligned} \alpha &\equiv \int_0^x du \int_{ky}^{\pi-y} |\Delta_y f| \frac{dv}{v} = \bar{o}(x), \\ \beta &\equiv \int_0^y dv \int_{kx}^{\pi-x} |\Delta_x f| \frac{du}{u} = \bar{o}(y), \end{aligned}$$

and (C_1).

From each of these tests the corresponding test for simple series can immediately be deduced; but it will be noticed that the most general continuity condition we use is (C_1) and not the analogue of $(\overline{C})$, namely:
(C) *the series* (1.21) *is summable, to sum s, by some Cesàro means.*
Thus we fail to extend completely to double series the tests $(\overline{HL})$ and $(\overline{L}_R)$, and we fail to obtain any analogue of Hardy and Littlewood's generalization $(\overline{Y}_{HL})$ of $(\overline{Y})$ other than (Y_P). The problem remains unsolved whether we can replace (C_1) by (C) in (L_R), (Y_P), and (HL). This problem, if one follows the ideas in simple series, involves proving that the characteristic† conditions of these tests imply the equivalence of (C_1) and (C), and this in turn involves obtaining a generalization to double series of Hardy and Littlewood's‡ theorem on summability and continuity. Of course the fact that each of the above tests leads directly to the corresponding test for simple series is due largely to the equivalence of $(\overline{C}_1)$ to $(\overline{C})$ whenever the latter is used.

To establish the above tests we deduce first (L_R) and then show that (L_R) contains all the other tests as particular cases. The facts in regard to (L_R) we state for convenience in the following theorem, the proof of which is to be found in §§2.0 to 3.1. The facts in regard to the relation of (L_R) to the rest of the tests are given in Theorems II to VII below.

THEOREM I. *If* (L_R) *holds, then the series* (1.21) *converges, to sum s.*

1.6. Turning now to the logical relations§ between the tests, we first state the following theorems, the proofs of which are to be found in §§4.1 to 13.1.

THEOREM II. (a) *The conditions* (L_1) *and* (L_2) *are equivalent.* (b) *The condition* (L_2) *implies* (L_P), *while* (L_P) *implies* (L_2) *if* (C_1^*) *holds.* (c) *The condition* (L_P) *implies* (L_R), *while* (L_R) *implies* (L_P) *if*

$$\phi_1^*(x, y) = O(xy). \tag{1.61}$$

Thus (L_1), (L_2), (L_P), *and* (L_R) *are equivalent if* (C_1^*) *holds.*‖

† The characteristic condition of a test consists of the conditions individual to the test. It is to be distinguished from the continuity condition, which is either (C_0) or a generalization of (C_0). In (L_R), for example, the characteristic condition is $(L_R')+(L_R'')$.

‡ Hardy and Littlewood, 7.

§ All our conclusions here are to the effect that certain conditions imply others. We make no attempt to prove that the implications stated are not reversible. For some examples of this type, see Hardy, 6. Hardy's examples are for simple series, but conclusions for double series can easily be deduced from them.

In discussing these relations it is well to point out again that, because one test is included in another, the latter is not for that reason a better test. If one reasoned in that way the best test would be the one in which the condition for convergence is that the series converge.

‖ This theorem contains some new information for simple series: that $(\overline{L}_P)$ implies $(\overline{L}_2)$ if (C_1^*) holds, and that $(\overline{L}_R)$ implies $(\overline{L}_P)$ if the condition corresponding to (1.61) holds.

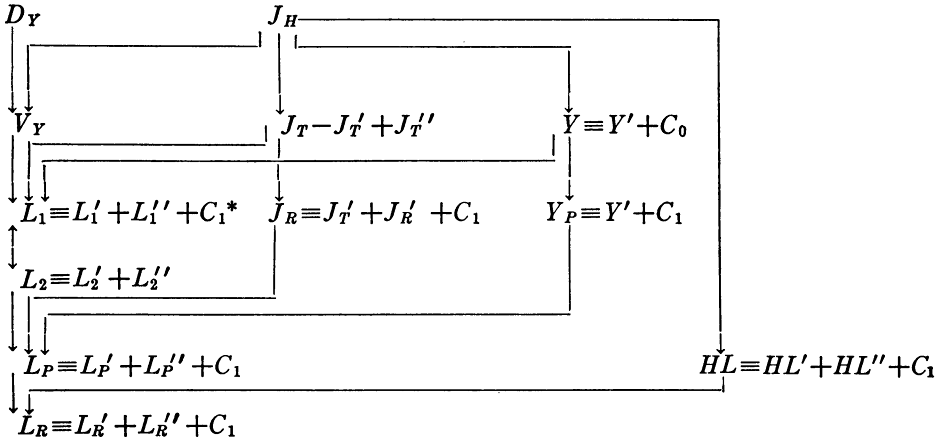
THEOREM III. *The condition* (J_T) *implies* (J_R) *and* (L_1), *both with* $s=f(+0, +0)$. *Moreover,* (J_R) *implies* (L_P).

THEOREM IV. *The condition* (Y) *implies* (Y_P) *and* (L_1), *while* (Y_P) *implies* (L_P).

THEOREM V. *The condition* (HL) *implies* (L_R).

THEOREM VI. *The condition* (J_H) *implies* (Y) *and* (HL), *both with* $s=f(+0, +0)$, *and the latter with* $p_1=p_2=p_3=1$.

THEOREM VII. *The condition* (V_Y) *implies* (L_1) *with* $s=F(+0, +0)$.



Combining these results with the fact that both (J_H) and (D_Y) imply (V_Y),† and (J_H) implies (J_T),‡ we obtain the accompanying diagram. In this diagram a directed line running from one letter to another indicates that the condition represented by the former implies that represented by the latter. In any implication in which s occurs only in the implied condition, s is understood to have the value indicated in the above theorem concerning this implication. It should be noted that, aside from the differences due to our use of (C_1) rather than (C), there are only two essential differences between this diagram and the author's§ diagram for simple series: first, we do not indicate here any implications between characteristic conditions; and secondly, we have here two new conditions, (J_T) and (J_R). In regard to the first difference it might be pointed out that our proofs show that all implications indicated for simple series carry over to double series, and thus that, in particular, the characteristic condition of (L_R) is implied by every other

† See the footnote on (D_Y) and Young, 15, p. 181.

‡ Tonelli, 13, p. 470.

§ Gergen, 4, p. 257.

characteristic condition. In regard to (J_T) and (J_R), these conditions, while analogous to $(\bar{J})$ in some respects, do not seem to be contained in (V_Y), (Y), and (HL). For this reason, and also because of their general character, these conditions seem to be essentially connected with a space of higher dimensionality than one.

2.0. **Lemmas for Theorem I.** The proof of Theorem I rests on the following lemmas. In these lemmas and throughout the rest of the paper we write for convenience

$$K(u, v; x, y) = K(u, v) = \sin\frac{\pi u}{x}\csc\frac{1}{2}u\sin\frac{\pi v}{y}\csc\frac{1}{2}v.$$

We shall always suppose that x, y, k are numbers such that

$$0 < x \leqq \pi,\ 0 < y \leqq \pi,\ 0 < k. \tag{2.01}$$

We understand by A a number whose value is independent of all or any group of the variables u, v, x, y, k with which we are concerned at the moment, for those values of the variables in question lying in the proper range. The range for u and that for v is always specified. The range for x either is completely specified or else it is understood to be that part of the range indicated in (2.01) not excluded by any partial specification. A similar understanding holds with regard to y and k.

We shall often have occasion to use the following formula for integrating by parts:

$$\begin{aligned}\int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho\psi'(u)\psi''(v)dv \\ = \rho_1(a_2, b_2)\psi(a_2,b_2) - \int_{a_1}^{a_2} \rho_1(u, b_2)\psi_u(u, b_2)du \\ - \int_{b_1}^{b_2} \rho_1(a_2, v)\psi_v(a_2, v)dv + \int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho_1\psi_{uv}dv,\end{aligned} \tag{2.02}$$

where

$$\psi(u, v) = \psi'(u)\psi''(v),$$

$$\rho_1(u, v) = \int_{a_1}^{u} d\sigma \int_{b_1}^{v} \rho(\sigma, t)dt.$$

This formula is valid if ρ is integrable on $(a_1, b_1; a_2, b_2)$, ψ' is absolutely continuous on (a_1, a_2), and ψ'' is absolutely continuous on (b_1, b_2).†

† This formula can readily be established by a double application of the formula for integrating by parts an integral involving but one variable and the application of other familiar results in the theory of Lebesgue integrals. The only question likely to occur is that of the measurability of the function $\int_{b_1}^{v}\rho(u, t)dt$ and this question is answered in a theorem of Carathéodory, 2, p. 656. In any case the formula is a particular case of one given by Hobson, 8, p. 666.

We have

$$s_{m,n} - s = \frac{1}{\pi^2}\int_0^\pi du \int_0^\pi \phi(u, v) K\{u, v; \pi/(m + \tfrac{1}{2}), \pi/(n + \tfrac{1}{2})\} dv,$$

and our problem is to show that, if (L_R) holds, then

$$S(x, y) \equiv \int_0^\pi du \int_0^\pi \phi K(u, v; x, y) = o(1).$$

As in the proof of $(\overline{L}_R)$ the problem is solved by breaking this integral into several parts and considering each part separately. In the lemmas we consider integrals over the region "near" the boundary of Q and also several functions which occur in the treatment of the integral over the area "away" from the boundary.

2.1. Lemma 1. *If* (C_1) *holds and if* $0<a$, $0<b$, *then*

$$I_1(x, y) \equiv \int_0^{ax} du \int_0^{by} \phi K dv = o(1). \tag{2.11}$$

Further,

$$I_2 \equiv \int_{\pi - ax}^{\pi} du \int_{\pi - by}^{\pi} \phi K dv = o(1)$$

under the sole assumption that ϕ *is integrable in* Q.

We have

$$\left.\begin{array}{ll} uv\,|\,K\,| < A, & uvx\,|\,K_u\,| < A \\ uvy\,|\,K_v\,| < A, & uvxy\,|\,K_{uv}\,| < A \end{array}\right\} \text{ for } 0 < u \leqq \pi,\ 0 < v \leqq \pi. \tag{2.12}$$

Further, applying (2.02),

$$I_1 = \phi_1(ax, by)K(ax, by) - \int_0^{ax} \phi_1(u, by)K_u(u, by)du$$

$$- \int_0^{by} \phi_1(ax, v)K_v(ax, v)dv + \int_0^{ax} du \int_0^{by} \phi_1 K_{uv} dv.$$

Thus

$$I_1 = O\left\{ \underset{0<u\leqq ax,\, 0<v\leqq by}{\text{maximum}} \left| \phi_1(u, v)/(uv) \right| \right\} = o(1)$$

by (C_1). This is (2.11).

As for I_2, we have immediately, since ϕ is integrable in Q,

$$I_2 = O\left\{ \int_{\pi-ax}^{\pi} du \int_{\pi-by}^{\pi} |\phi|\, dv \right\} = o(1).$$

2.2. LEMMA 2. *If* (C_1) *and* (L_R'') *hold, then*

$$\int_0^x du \int_{\pi-y}^{\pi} \phi dv = o(x), \tag{2.21}$$

$$|\phi_1(x, y)| < Axy. \tag{2.22}$$

We observe first that, corresponding to an arbitrary number $0 < \epsilon$, we can choose $0 < k_0$ and $0 < \delta < \pi/2$ so that

$$|\phi_1(x, y)| < \epsilon xy \text{ for } x \leqq \delta,\ y \leqq \delta, \tag{2.23}$$
$$\alpha(x, y; k_0) < \epsilon x, \quad \beta(x, y; k_0) < \epsilon y \text{ for } x \leqq \delta,\ y \leqq \delta.$$

We observe next that, if x, c, and z are numbers such that

$$x \leqq \delta,\ (k_0 + 1)z \leqq \delta \leqq c < c + z \leqq \pi, \tag{2.24}$$

we consequently have

$$\begin{aligned}\left| \int_0^x du \int_c^{c+z} \phi dv \right| &= \left| \int_0^x du \left\{ \int_{(k_0+1)z}^{c+z} - \int_{k_0 z}^{c} + \int_{k_0 z}^{(k_0+1)z} \right\} \phi dv \right| \\ &\leqq \int_0^x du \int_{k_0 z}^{\pi - z} |\phi(u, v + z) - \phi(u, v)|\, dv + |\phi_1\{x, (k_0 + 1)z\}| \\ &\quad + |\phi_1(x, k_0 z)| \\ &< \pi\epsilon x + \epsilon x(k_0 + 1)z + \epsilon x k_0 z < 2\pi\epsilon x.\end{aligned} \tag{2.25}$$

We can now easily prove that (2.21) holds. In fact, if $x \leqq \delta$ and $(k_0+1)y \leqq \delta$, then (2.24) holds with $c = \pi - y$ and $z = y$; and hence, applying (2.25),

$$\left| \int_0^x du \int_{\pi-y}^{\pi} \phi\, dv \right| < 2\pi\epsilon x.$$

Since ϵ was arbitrary, this proves that (2.21) holds.

As for (2.22), let us first suppose that $x \leqq \delta < y$. Then, choosing z so that $N = (\pi - \delta)/z$ is an integer and so that $0 < (k_0+1)z \leqq \delta$, and denoting by a the largest of the numbers $\delta, \delta + z, \cdots, \pi - z$ less than y, we have

$$\begin{aligned}|\phi_1(x, y)| &\leqq |\phi_1(x, \delta)| + \sum_{n=0}^{N-1} \left| \int_0^x du \int_{\delta+nz}^{\delta+(n+1)z} \phi\, dv \right| + \left| \int_0^x du \int_a^y \phi dv \right| \\ &< \epsilon x\delta + 2\pi N\epsilon x + 2\pi\epsilon x\end{aligned}$$

upon applying (2.23) and (2.25). Thus, for $x \leqq \delta < y$, we have

$$|\phi_1(x, y)| < Axy. \tag{2.26}$$

Similarly, (2.26) holds for $y \leqq \delta < x$. Accordingly, because of (2.23) and the obvious fact that (2.26) holds for $\delta \leqq x$, $\delta \leqq y$, the proof is complete.

2.3. LEMMA 3. *If* (C_1) *and* (L_R'') *hold and if* $0 < a$, $0 < b$, *then*

$$I_3 \equiv \int_0^{ax} du \int_{\pi - by}^{\pi} \phi K \, dv = o(1). \tag{2.31}$$

Writing

$$\bar{\phi}(u, v) = \int_0^u d\sigma \int_{\pi - by}^{v} \phi(\sigma, t) dt,$$

we have

$$\begin{aligned} I_3 &= \bar{\phi}(ax, \pi) K(ax, \pi) - \int_0^{ax} \bar{\phi}(u, \pi) K_u(u, \pi) du \\ &\quad - \int_{\pi - by}^{\pi} \bar{\phi}(ax, v) K_v(ax, v) dv + \int_0^{ax} du \int_{\pi - by}^{\pi} \bar{\phi} K_{uv} dv \\ &= O(\text{maximum} \, | \bar{\phi}(u, v)/v |) \qquad (0 < u \leqq ax, \pi - by \leqq v \leqq \pi). \end{aligned}$$

Using Lemma 2 now we conclude the proof.

2.4. LEMMA 4. *If* (C_1) *and* (L_R'') *hold and if* $0 < a$, *then*

$$J_1(x, y; k) \equiv \int_0^{ax} du \int_{ky}^{\pi - 2y} \phi \Omega \, dv = \bar{o}(1),$$

where

$$\Omega = \Omega(u, v; x, y) = \Omega(u, v) = \sin \frac{\pi u}{x} \csc \frac{1}{2} u \sin \frac{\pi v}{y} \omega(v; y),$$

and

$$\omega(v; y) = 2 \csc \tfrac{1}{2}(v + y) - \csc \tfrac{1}{2}(v + 2y) - \csc \tfrac{1}{2} v.$$

Writing

$$\bar{\phi}(u, v) = \int_0^u d\sigma \int_{ky}^{v} \phi(\sigma, t) dt,$$

we have, by Lemma 2,

$$|\bar{\phi}(u, v)| < Auv \quad \text{for } 0 < u \leqq \pi, \ ky \leqq v \leqq \pi.$$

On the other hand, we have

(2.41) $$v^3|\omega(v;y)| < Ay^2, \qquad v^4|\omega_v| < Ay^2 \text{ for } y < v \leqq \pi - y;\dagger$$

and thus

(2.42) $$\left.\begin{aligned} uv^3|\Omega(u,v)| < Ay^2,\ uv^3x|\Omega_u| < Ay^2 \\ uv^3|\Omega_v| < Ay,\ uv^3x|\Omega_{uv}| < Ay \end{aligned}\right\}$$

for $0 < u \leqq \pi,\ y \leqq v \leqq \pi - y.$

Thus

$$\begin{aligned} J_1 &= \bar{\phi}(ax, \pi - 2y)\Omega(ax, \pi - 2y) - \int_0^{ax} \bar{\phi}(u, \pi - 2y)\Omega_u(u, \pi - 2y)du \\ &\quad - \int_{ky}^{\pi-2y} \bar{\phi}(ax, v)\Omega_v(ax, v)dv + \int_0^{ax} du \int_{ky}^{\pi-2y} \bar{\phi}\Omega_{uv}dv \\ &= \bar{O}(y^2 + y^2 + 1/k + 1/k) = \bar{o}(1); \end{aligned}$$

which proves the lemma.

2.5. Lemma 5. *If* (C_1) *and* (L_K'') *hold and if* $0<a$, *then*

(2.51) $$I_4 \equiv \int_0^{ax} du \int_0^{\pi} \phi K\, dv = o(1).$$

We have, by Lemmas 1, 3, and 4,

$$\begin{aligned} 4I_4 &= \int_0^{ax} du \left\{ \int_{ky}^{\pi-2y} + 2\int_{(k+1)y}^{\pi-y} + \int_{(k+2)y}^{\pi} \right\} \phi K\, dv + \bar{o}(1) \\ &= -J_1 + J_1' + \bar{o}(1) = J_1' + \bar{o}(1), \end{aligned}$$

where

$$J_1' = \int_0^{ax} \frac{\sin \dfrac{\pi u}{x}}{\sin \frac{1}{2}u} du \int_{ky}^{\pi-2y} \left\{ \frac{\Delta_{2y}\phi}{\sin\frac{1}{2}(v+2y)} - \frac{2\Delta_y \phi}{\sin \frac{1}{2}(v+y)} \right\} \sin \frac{\pi v}{y} dv.$$

But

$$\begin{aligned} J_1' &= \bar{O}\left\{ \frac{1}{x} \int_0^{ax} du \int_{ky}^{\pi-2y} \left| \frac{\Delta_{2y}\phi}{\sin\frac{1}{2}(v+2y)} - \frac{2\Delta_y\phi}{\sin\frac{1}{2}(v+y)} \right| dv \right\} \\ &= \bar{O}\{\alpha(ax, 2y; \tfrac{1}{2}k)/x + \alpha(ax, y; k)/x\} = \bar{o}(1). \end{aligned}$$

Since I_4 is independent of k, the lemma follows.

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† Gergen, 4, p. 271. The first inequality in (2.41) is (8.72) and the second is (8.73) of that paper with $M=A$, $m=2$, $\nu=1$, $\rho=0$, $t=v$.

2.6. LEMMA 6. *If* (L_R) *and* (L_R'') *hold, then*

$$J_2 \equiv \int_{\pi-y}^{\pi} dv \int_{kx}^{\pi-x} |\Delta_x\phi| \frac{du}{u} = \bar{o}(1) \tag{2.61}$$

and $0 < x_0 < \pi$ *and* $0 < k_0$ *can be found so that*

$$\beta(x, y; k) < Ay \qquad \text{for } x \leqq x_0,\ k_0 \leqq k. \tag{2.62}$$

We first observe that, corresponding to an arbitrary number $0 < \epsilon$, we can choose $0 < k_0$ and x_0 so that

$$0 < x_0(k_0 + 1) < \pi/2, \tag{2.63}$$

$$\beta(x, y; k_0) < \epsilon y, \qquad \gamma(x, y; k_0) < \epsilon \text{ for } x \leqq x_0,\ y \leqq x_0. \tag{2.64}$$

We next observe that, if

$$x \leqq x_0,\ (k_0 + 1)z \leqq x_0 \leqq c < c + z \leqq \pi, \tag{2.65}$$

we consequently have

$$\begin{aligned} \int_c^{c+z} dv \int_{k_0 x}^{\pi-x} |\Delta_x\phi| \frac{du}{u} &= \left\{ \int_{(k_0+1)z}^{c+z} - \int_{k_0 z}^{c} + \int_{k_0 z}^{(k_0+1)z} \right\} \int_{k_0 x}^{\pi-x} |\Delta_x\phi| \frac{du}{u} \\ &\leqq \pi\gamma(x, z; k_0) + \beta\{x, (k_0 + 1)z; k_0\} \\ &< \pi\epsilon + \epsilon(k_0 + 1)z < 2\pi\epsilon. \end{aligned} \tag{2.66}$$

Consider, then, (2.61). If $x \leqq x_0$ and $(k_0+1)y \leqq x_0$, then (2.65) holds with $c = \pi - y$ and $z = y$. Accordingly, because of (2.66) and the fact that

$$J_2(x, y; k) \leqq J_2(x, y; k_0) \text{ for } k_0 \leqq k,$$

we have

$$J_2 < 2\pi\epsilon \text{ for } x \leqq x_0,\ (k_0 + 1)y \leqq x_0,\ k_0 \leqq k.$$

Since ϵ was arbitrary, this proves that (2.61) holds.

As for (2.62), we observe that, because of (2.64) and the fact that

$$\beta(x, y; k) \leqq \beta(x, y; k_0) \text{ for } k_0 \leqq k,$$

it is enough to prove that

$$\beta(x, y; k_0) < Ay \text{ for } x \leqq x_0 < y.$$

But this is immediate; for, choosing z so that $N = (\pi - x_0)/z$ is an integer and so that $0 < (k_0+1)z \leqq x_0$, we have

$$\beta(x, y; k_0) \leqq \beta(x, \pi; k_0) \leqq \beta(x, x_0; k_0) + \sum_{n=0}^{N-1} \int_{x_0+nx}^{x_0+(n+1)x} dv \int_{k_0x}^{\pi-x} |\Delta_x\phi| \frac{du}{u}$$
$$< \epsilon x_0 + 2\pi N\epsilon < Ay$$

for $x \leqq x_0 < y$ by (2.64) and (2.66). This completes the proof.

2.7. LEMMA 7. *If* (L_R) *holds and if* $0 < b$, *then*

$$I_5 = \int_{\pi-by}^{\pi} dv \int_0^{\pi} \phi K\, du = o(1).$$

We have, using Lemmas 1 and 3,

$$4I_5 = \int_{\pi-by}^{\pi} dv \left\{ \int_{kx}^{\pi-2x} + 2\int_{(k+1)x}^{\pi-x} + \int_{(k+2)x}^{\pi} \right\} \phi K\, du + \bar{o}(1)$$
$$= J_3' + J_3'' + \bar{o}(1),$$

where

$$J_3' = \int_{\pi-by}^{\pi} \frac{\sin \dfrac{\pi v}{y}}{\sin \frac{1}{2}v} dv \int_{kx}^{\pi-2x} \left\{ \frac{\Delta_{2x}\phi}{\sin \frac{1}{2}(u+2x)} - \frac{2\Delta_x\phi}{\sin \frac{1}{2}(u+x)} \right\} \sin \frac{\pi u}{x}\, du,$$
$$J_3'' = -\int_{\pi-by}^{\pi} dv \int_{kx}^{\pi-x} \phi\, \Omega(v, u; y, x) du.$$

Now, as for J_3', we have immediately, upon applying Lemma 6,

$$J_3' = \bar{O}\{J_2(2x, by; \tfrac{1}{2}k) + J_2(x, by; k)\} = \bar{o}(1).$$

Finally, as for J_3'', upon setting

$$\bar{\phi}(u, v) = \int_{\pi-by}^{v} dt \int_{kx}^{u} \phi(\sigma, t) d\sigma,$$

we have

$$|\bar{\phi}(u, v)| < Au \text{ for } kx \leqq u \leqq \pi,\ 0 < \pi - by \leqq v \leqq \pi,$$

by Lemma 2. Thus, noting that (2.42) is applicable to the function

$$\bar{\Omega}(u, v) = \Omega(v, u; y, x),$$

we have

$$J_3'' = -\bar{\phi}(\pi - 2x, \pi)\bar{\Omega}(\pi - 2x, \pi) + \int_{kx}^{\pi-2x} \bar{\phi}(u, \pi)\bar{\Omega}_u(u, \pi)du$$

$$+ \int_{\pi-by}^{\pi} \bar{\phi}(\pi - 2x, v)\bar{\Omega}_v(\pi - 2x, v)dv - \int_{\pi-by}^{\pi} dv \int_{kx}^{\pi-2x} \bar{\phi}\,\bar{\Omega}_{uv}\,du$$

$$= \bar{O}(x^2 + 1/k + x^2 + 1/k) = \bar{o}(1).$$

This completes the proof.

2.8. Lemma 8. *If* (C_1) *and* (L_R'') *hold, then*

$$J_3 \equiv \int_{kx}^{\pi-2x} \omega(u; x) \sin \frac{\pi u}{x} du \int_{ky}^{\pi-2y} \omega(v; y)\phi(u + x, v + y) \sin \frac{\pi v}{y} dv = \bar{o}(1).$$

First, setting

$$w(u, v) = \omega(u; x) \sin (\pi u/x)\omega(v; y) \sin (\pi v/y),$$

and using (2.41), we have

$$\left.\begin{array}{ll} u^3v^3 |\, w \,| < Ax^2y^2, & u^3v^3 |\, w_u \,| < Axy^2 \\ u^3v^3 |\, w_v \,| < Ax^2y, & u^3v^3 |\, w_{uv} \,| < Axy \end{array}\right\} \text{ for } x \leqq u \leqq \pi - x,\ y \leqq v \leqq \pi - y.$$

Next, setting

$$\bar{\phi}(u, v) = \int_{kx}^{u} d\sigma \int_{ky}^{v} \phi(\sigma + x, t + y)dt,$$

and using Lemma 2, we have

$$|\,\bar{\phi}(u, v)\,| < Auv \text{ for } kx \leqq u \leqq \pi - x,\ ky \leqq v \leqq \pi - y,\ 1 \leqq k.$$

Thus

$$J_3 = \bar{\phi}(\pi - 2x, \pi - 2y)w(\pi - 2x, \pi - 2y) - \int_{kx}^{\pi-2x} \bar{\phi}(u, \pi - 2y)w_u(u, \pi - 2y)du$$

$$- \int_{ky}^{\pi-2y} \bar{\phi}(\pi - 2x, v)w_v(\pi - 2x, v)dv + \int_{kx}^{\pi-2x} du \int_{ky}^{\pi-2y} \bar{\phi} w_{uv}\,dv$$

$$= \bar{O}(x^2y^2 + y^2/k + x^2/k + 1/k^2) = \bar{o}(1).$$

This proves the lemma.

2.9. Lemma 9. *If* (L_R') *and* (L_R'') *hold, then*

$$J_4 \equiv \int_{ky}^{\pi-2y} |\,\omega(v; y)\,|\, dv \int_{kx}^{\pi-x} |\,\Delta_x\phi(u, v + y)\,| \frac{du}{u} = \bar{o}(1).$$

We have

$$J_4 = \bar{O}\left\{y^2 \int_{ky}^{\pi-2y} \frac{dv}{v^3} \int_{kx}^{\pi-x} |\Delta_x \phi(u, v + y)| \frac{du}{u}\right\}$$

by (2.41). Thus, integrating by parts and using Lemma 6,

$$J_4 = \bar{O}\left\{y^2\beta(x, \pi; k) + y^2 \int_{ky}^{\pi} \beta(x, v; k)\frac{dv}{v^4}\right\} = \bar{o}(1).$$

3.1. **Proof of Theorem I.** Because of the symmetry of the condition (L_R) with respect to the arguments of f, it is plain that Lemmas 3, 5, and 7 hold if we interchange the arguments of f in the integrals appearing there. Thus, if $0<a$, $0<b$, then

$$\int_0^{by} dv \int_{\pi-ax}^{\pi} \phi(u, v)K(u, v; x, y)du = o(1),$$

$$\int_0^{by} du \int_0^{\pi} \phi K du = o(1), \quad \int_{\pi-ax}^{\pi} du \int_0^{\pi} \phi K\, dv = o(1).$$

Now it is readily seen by these relations and Lemmas 1, 3, 5, and 7 that

$$16S(x, y) = \left\{\int_{kx}^{\pi-2x} + 2\int_{(k+1)x}^{\pi-x} + \int_{(k+2)x}^{\pi}\right\} G(u; x, y; k)du + \bar{o}(1),$$

where

$$G = \left\{\int_{ky}^{\pi-2y} + 2\int_{(k+1)y}^{\pi-y} + \int_{(k+2)y}^{\pi}\right\} \phi K\, dv.$$

Hence, making in each of these integrals a change of variables which carries the region of integration into $(kx, ky, \pi-2x, \pi-2y)$, and collecting the terms properly, we have

$$16S = \int_{kx}^{\pi-2x} \sin\frac{\pi u}{x} du \int_{ky}^{\pi-2y} \Psi(u, v; x, y) \sin\frac{\pi v}{y} dv + \bar{o}(1),$$

where

$$\begin{aligned}\Psi = &\left\{\frac{\Delta_{x,y}\phi(u + x, v + y)}{\sin\frac{1}{2}(u + 2x)\sin\frac{1}{2}(v + 2y)} - \frac{\Delta_{x,y}\phi(u, v + y)}{\sin\frac{1}{2}u \sin\frac{1}{2}(v + 2y)} - \frac{\Delta_{x,y}\phi(u + x, v)}{\sin\frac{1}{2}(u + 2x)\sin\frac{1}{2}v}\right.\\ &\left. - \frac{\Delta_{x,y}\phi(u, v)}{\sin\frac{1}{2}u \sin\frac{1}{2}v}\right\} - \omega(v, y)\left\{\frac{\Delta_x\phi(u + x, v + y)}{\sin\frac{1}{2}(u + 2x)} - \frac{\Delta_x\phi(u, v + y)}{\sin\frac{1}{2}u}\right\}\\ &- \omega(u, x)\left\{\frac{\Delta_y\phi(u + x, v + y)}{\sin\frac{1}{2}(v + 2y)} - \frac{\Delta_y\phi(u + x, v)}{\sin\frac{1}{2}v}\right\}\\ &+ \omega(u; x)\omega(v; y)\phi(u + x, v + y)\\ = &\ S_1 + S_2 + S_3 + S_4, \text{ say.}\end{aligned}$$

But

$$\int_{kx}^{\pi-2x} \sin\frac{\pi u}{x} du \int_{ky}^{\pi-2y} S_1 \sin\frac{\pi v}{y} dv = \bar{O}\left\{ \int_{kx}^{\pi-2x} du \int_{ky}^{\pi-2y} | S_1 | \, dv \right\}$$
$$= \bar{O}\{\gamma(x, y; k)\} = \bar{o}(1)$$

and, by Lemma 9,

$$\int_{kx}^{\pi-2x} \sin\frac{\pi u}{x} du \int_{ky}^{\pi-2y} S_2 \sin\frac{\pi v}{y} dv$$
$$= \bar{O}\left\{ \int_{ky}^{\pi-2y} | \omega(v; y) | \, dv \int_{kx}^{\pi-2x} | \Delta_x \phi(u, v+y) | \frac{du}{u} \right\} = \bar{o}(1).$$

Similarly,

$$\int_{kx}^{\pi-2x} \sin\frac{\pi u}{x} du \int_{ky}^{\pi-2y} S_3 \sin\frac{\pi v}{y} dv = \bar{o}(1).$$

Finally,

$$\int_{kx}^{\pi-2x} \sin\frac{\pi u}{x} du \int_{ky}^{\pi-2y} S_4 \sin\frac{\pi v}{y} dv = \bar{o}(1)$$

by Lemma 8.

Thus,

$$s(x, y) = \bar{o}(1)$$

if (L_R) holds. This proves the theorem, since S is independent of k.

4.1. **Lemmas for Theorem II.** LEMMA 1. *If* (L_P') *and* (L_P'') *hold, then*

$$J_5 \equiv \int_{\pi-y}^{\pi} dv \int_{kx}^{\pi-x} \left| \Delta_x\left\{\frac{\phi}{u}\right\}\right| du = \bar{o}(1), \tag{4.11}$$

and $0 < x_0 < \pi$ *and* $0 < k_0$ *can be found so that*

$$\eta(x, y; k) < Ay \text{ for } x \leqq x_0,\ k_0 \leqq k. \tag{4.12}$$

Further, if (L_2) *holds, then*

$$J_5(x, y; 1) = o(1). \tag{4.13}$$

The proof concerning (4.11) and (4.12) is much the same as that of Lemma 6, §2.6. Given $0 < \epsilon$ we can, because of (L_P') and (L_P''), choose $1 < k_0$ and x_0 so that (2.63) holds, and

$$\eta(x, y; k_0) < \epsilon y,\ \zeta(x, y; k_0) < \epsilon \text{ for } x \leqq x_0,\ y \leqq x_0. \tag{4.14}$$

Thus, if $x \leqq x_0$, $(k_0+1)z \leqq x_0 \leqq c < c+z \leqq \pi$, we have

$$\int_c^{c+z} dv \int_{k_0 z}^{\pi-x} \left|\Delta_x\left\{\frac{\phi}{u}\right\}\right| du \leqq \pi \int_c^{c+z} \frac{dv}{v} \int_{k_0 x}^{\pi-x} \left|\Delta_x\left\{\frac{\phi}{u}\right\}\right| du$$

$$= \pi\left\{ \int_{(k_0+1)z}^{c+z} - \int_{k_0 z}^{c} + \int_{k_0 z}^{(k_0+1)z} \right\} \frac{dv}{v} \int_{k_0 x}^{\pi-x} \left|\Delta_x\left\{\frac{\phi}{u}\right\}\right| du$$

$$\leqq \pi\zeta(x, z; k_0) + \pi\eta\{x, (k_0+1)z; k_0\}/(k_0 z)$$

$$< \pi\epsilon + \pi\epsilon(1 + 1/k_0) < 3\pi\epsilon.$$

Choosing suitable sets of values for c and z in this inequality, and making use of (4.14) and the fact that

$$J_5(x, y; k) \leqq J_5(x, y; k_0), \quad \eta(x, y; k) \leqq \eta(x, y; k_0) \text{ for } k_0 \leqq k,$$

we deduce easily (4.11) and (4.12).

As for (4.13), we have

$$J_5(x, y; 1) = O\left[\left\{ \int_{2y}^{\pi} - \int_{y}^{\pi-y} + \int_{y}^{2y} \right\} \frac{dv}{v} \int_{x}^{\pi-x} \left|\Delta_x\left\{\frac{\phi}{u}\right\}\right| du\right]$$

$$= O\{\zeta_1(x, y) + \eta_1(x, 2y)/y\} = o(1)$$

by (L_2). This completes the proof.

4.2. Lemma 2. *If* (L_P') *and* (L_P'') *hold, then* (1.61) *holds. Moreover, if* (L_2) *holds, then* (C_1^*) *holds.*

Let $\lambda(x, y; k)$ be the upper bound of

$$\xi(u, v; k)/u + J_5(x, y; k)$$

for a fixed k for $0 < u \leqq x$, $0 < v \leqq y$, and let

$$x_\mu = x\{k/(k+1)\}^\mu, \quad y_\mu = y\{k/(k+1)\}^\mu$$

for $\mu = 0, 1, \cdots$. Then, for $(k+1)x \leqq \pi$, $(k+1)y \leqq \pi$, $1 \leqq k$,

$$\phi_1^*(kx, ky) \leqq k \sum_{\mu=0}^{\infty} \left\{ y_\mu \int_0^{kx} du \int_{ky_{\mu+1}}^{ky_\mu} |\phi| \frac{dv}{v} \right\}$$

$$= k \sum_{\mu=0}^{\infty} \left[y_\mu \int_0^{kx} du \left\{ \int_{ky_{\mu+1}}^{\pi-y_{\mu+1}} - \int_{ky_\mu}^{\pi} + \int_{\pi-y_{\mu+1}}^{\pi} \right\} |\phi| \frac{dv}{v} \right]$$

$$\leqq k \sum_{\mu=0}^{\infty} \left[y_\mu \left\{ \xi(kx, y_{\mu+1}; k) + \int_0^{kx} du \int_{\pi-y}^{\pi} |\phi| \frac{dv}{v} \right\} \right]$$

$$\leqq (k+1)^3 xy\lambda(kx, y; k) + (k+1)^2 y \int_0^{kx} du \int_{\pi-y}^{\pi} |\phi|\, dv,$$

and

$$\int_0^{kx} du \int_{\pi-y}^{\pi} |\phi|\, dv \leqq k \sum_{\mu=0}^{\infty} \left\{ x_\mu \int_{\pi-y}^{\pi} dv \int_{kx_{\mu+1}}^{kx_\mu} |\phi| \frac{du}{u} \right\}$$
$$\leqq k \sum_{\mu=0}^{\infty} \left[x_\mu \left\{ J_5(x_{\mu+1}, y; k) + \int_{\pi-y}^{\pi} dv \int_{\pi-x}^{\pi} |\phi| \frac{du}{u} \right\} \right]$$
$$\leqq (k+1)^2 x\lambda(kx, y; k) + (k+1)^2 o(x).$$

Accordingly,

$$\phi_1^*(kx, ky) \leqq (k+1)^4 xy \{2\lambda(kx, y; k) + o(1)\}.$$

Now, if (L_P') and (L_P'') hold, then, using Lemma 1,

$$\lambda(kx, y; k) = O(1)$$

for some fixed k; while if (L_2) holds, then

$$\lambda(x, y; 1) = o(1).$$

The lemma follows.

4.3. LEMMA 3. *If* (L_P') *and* (L_P'') *hold, or if* (L_R'), (L_R''), *and* (1.61) *hold, then*

(4.31) $$\phi_1^*(x, y) < Axy.$$

The proof concerning (L_P') and (L_P''), as well as that concerning (L_R') and (L_R''), closely resembles the proof of (2.22). We need consider only (L_P') and (L_P''). We first observe that, by (L_P'), (L_P''), and Lemma 2, numbers $0<\epsilon$, $1<k_0$, and $0<\delta<\pi/2$ can be found such that

$$\phi_1^*(x, y) < \epsilon xy, \quad \xi(x, y; k_0) < \epsilon x, \quad \eta(x, y; k_0) < \epsilon y$$

for $x \leqq \delta$, $y \leqq \delta$. We next observe that, if $x \leqq \delta$, $(k_0+1)z \leqq \delta \leqq c < c+z \leqq \pi$, we thus have

$$\int_0^x du \int_c^{c+z} |\phi|\, dv \leqq \int_0^x du \left\{ \int_{(k_0+1)z}^{c+z} - \int_{k_0 z}^{c} + \int_{k_0 z}^{(k_0+1)z} \right\} |\phi| \frac{dv}{v}$$
$$\leqq \pi\xi(x, y; k_0) + \phi_1^*\{x, (k_0+1)z\}/(k_0 z)$$
$$< \pi\epsilon x + \epsilon x(1 + 1/k_0) < 3\pi\epsilon x.$$

Proceeding now as in the proof of (2.22), we deduce (4.31).

4.4. LEMMA 4. *If* (4.31) *holds, then*

$$J_6 \equiv xy \int_{kx}^{\pi} \frac{du}{u^2} \int_{ky}^{\pi} \frac{|\phi|}{v^2}\, dv = \bar{o}(1).$$

We have, upon integrating by parts,

$$J_6 = \bar{O}\left\{xy\phi_1^*(\pi,\pi) + xy\int_{kx}^{\pi}\frac{\phi_1^*(u,\pi)}{u^3}du\right.$$

$$\left. + xy\int_{ky}^{\pi}\frac{\phi_1^*(\pi,v)}{v^3}dv + xy\int_{kx}^{\pi}\frac{du}{u^3}\int_{ky}^{\pi}\frac{\phi_1^*}{v^3}dv\right\}$$

$$= \bar{O}(xy + y/k + x/k + 1/k^2) = \bar{o}(1).$$

4.5. LEMMA 5. *If* (4.31) *holds, then* (L_R'') *is equivalent to* (L_P'').

We need consider only α and ξ. We have

$$\alpha - \xi = \bar{O}\left\{y\int_0^x du\int_{(k+1)y}^{\pi}\frac{|\phi|\,dv}{(v-y)v}\right\}$$

$$= \bar{O}\left\{y\phi_1^*(x,\pi) + y\int_{(k+1)y}^{\pi}\frac{\phi_1^*(x,v)}{(v-y)^2v}dv\right\}$$

$$= \bar{O}(xy + x/k) = \bar{o}(x);$$

and this proves the lemma.

4.6. LEMMA 6. *If* (L_R') *and* (L_R'') *hold, then*

$$J_7 \equiv y\int_{ky}^{\pi}\frac{dv}{v^2}\int_{kx}^{\pi-x}|\Delta_x\phi|\frac{du}{u} = \bar{o}(1).$$

Moreover, if (L_P') *and* (L_P'') *hold, then*

$$J_8 \equiv y\int_{ky}^{\pi}\frac{dv}{v^2}\int_{kx}^{\pi-x}\left|\Delta_x\left\{\frac{\phi}{u}\right\}\right|du = \bar{o}(1).$$

We may confine our attention to the second part of this lemma. We have

$$J_8 = \bar{O}\left\{y\eta(x,\pi;k) + y\int_{ky}^{\pi}\eta(x,v;k)\frac{dv}{v^3}\right\}$$

$$= \bar{O}(y + 1/k) = \bar{o}(1)$$

by (L_P') and (L_P'') and Lemma 1.

4.7. LEMMA 7. *If* (C_1^*) *holds, then* (L_R) *implies* (L_1), *and* (L_P) *implies* (L_2).

We may confine ourselves to (L_P) and (L_2). We have

$$\xi_1 = \int_0^x du\left\{\int_y^{ky} + \int_{ky}^{\pi-y}\right\}\left|\Delta_y\left\{\frac{\phi}{v}\right\}\right|dv$$

$$= \bar{O}[\phi_1^*\{x,(k+1)y\}/y + \xi(x,y;k)] = \bar{o}(x)$$

by (C_1^*) and (L_P). Thus, since ξ_1 is independent of k,

$$\xi_1 = o(x).$$

Similarly,

$$\eta_1 = o(y).$$

Accordingly, (L_2'') holds.

As for (L_2'), we have

$$\zeta_1 = \left\{ \int_x^{kx} \int_y^{\pi-y} + \int_x^{\pi-x} \int_y^{ky} - \int_x^{kx} \int_y^{ky} + \int_{kx}^{\pi-x} \int_{ky}^{\pi-y} \right\} \left| \Delta_{x,y} \left\{ \frac{\phi}{uv} \right\} \right| du\, dv$$

$$= O[\xi_1\{(k+1)x, y\}/x + \eta_1\{x, (k+1)y\}/y + \zeta] = \bar{o}(1)$$

by (C_1^*), (L_P), and (L_2''). The lemma follows.

5.1. **Proof of Theorem II.** We first note the identities

$$\frac{\Delta_{x,y}\phi}{uv} = \frac{xy}{u^2v^2}\phi + \left(1 + \frac{x}{u}\right)\frac{y}{v^2}\Delta_x\left\{\frac{\phi}{u}\right\}$$

$$+ \frac{x}{u^2}\left(1 + \frac{y}{v}\right)\Delta_y\left\{\frac{\phi}{v}\right\} + \left(1 + \frac{x}{u}\right)\left(1 + \frac{y}{v}\right)\Delta_{x,y}\left\{\frac{\phi}{uv}\right\},$$

$$\Delta_{x,y}\left\{\frac{\phi}{uv}\right\} = \frac{xy\phi}{u(u+x)v(v+y)} - \frac{y\Delta_x\phi}{(u+x)v(v+y)} - \frac{x\Delta_y\phi}{u(u+x)(v+y)}$$

$$+ \frac{\Delta_{x,y}\phi}{(u+x)(v+y)}.$$

We next note that it follows from these identities that

(5.11) $\gamma \leqq J_6 + 2J_8 + 2J_9 + 4\zeta$ for $(k+1)x \leqq \pi$, $(k+1)y \leqq \pi$, $1 \leqq k$,

where

$$J_9 = x\int_{kx}^{\pi} \frac{du}{u^2} \int_{ky}^{\pi-y} \left| \Delta_y \left\{ \frac{\phi}{v} \right\} \right| dv,$$

and that

(5.12) $\zeta \leqq J_6 + J_7 + J_{10} + \gamma$ for $(k+1)x \leqq \pi$, $(k+1)y \leqq \pi$,

where

$$J_{10} = x\int_{kx}^{\pi} \frac{du}{u^2} \int_{ky}^{\pi-y} \left| \Delta_y\phi \right| \frac{dv}{v}.$$

Consider, then, the first part of (c). If (L_P) holds, then (4.31) holds by Lemma 3, and accordingly, by Lemma 4,

$$J_6 = \bar{o}(1).$$

Further,

$$J_8 = \bar{o}(1)$$

by Lemma 6; and plainly, by the same reasoning as in Lemma 6,

$$J_9 = \bar{o}(1).$$

Thus, by (5.11), (L_P) implies (L_R'). But, by Lemmas 3 and 5, (L_P) also implies (L_R''). The truth of the first part of (c) follows.

Now consider the second part of (c). If (L_R) and (1.61) hold, then (4.31) holds by Lemma 3, and accordingly, by Lemma 4,

$$J_6 = \bar{o}(1).$$

Further,

$$J_7 = \bar{o}(1)$$

by Lemma 6, and plainly,

$$J_{10} = \bar{o}(1).$$

Thus, by (5.12), (L_R) and (1.61) imply (L_P'). But, by Lemmas 3 and 5, (L_R) and (1.61) also imply (L_P''). The second part of (c) follows.

As for (b), the first part of (b) is trivial. The second part is proved in Lemma 7.

Turning, finally, to (a), let us first suppose that (L_1) holds. Then, plainly (L_R) holds, and thus, by (c), since (L_1) contains (C_1^*), (L_P) also holds. Accordingly, (L_2) holds by Lemma 7. Thus (L_1) implies (L_2).

Suppose, on the other hand, that (L_2) holds. Then (L_P) holds by (b), and accordingly, (L_R) holds by (c). But (L_2) also implies (C_1^*) by Lemma 2. Thus, by Lemma 7 again, (L_1) holds. This completes the proof.

6.1. **Lemmas for Theorem III.** LEMMA 1. *If (J_R) holds, then*

$$f(x, y) = O(1). \tag{6.11}$$

We choose $0 < \epsilon$ and $0 < \delta < \pi/2$ so that

$$W_1(x, y) < \epsilon, \ W_2(x, y) < \epsilon \text{ for } x \leqq 2\delta, \ y \leqq 2\delta. \tag{6.12}$$

Then we have, since $f(\delta, \delta)$ is finite,

$$\begin{aligned} |f(x, y)| &\leqq |f(x, y) - f(\delta, y)| + |f(\delta, y) - f(\delta, \delta)| + |f(\delta, \delta)| \\ &\leqq W_1(\delta, y) + W_2(\delta, \delta) + |f(\delta, \delta)| < A \end{aligned}$$

for $x \leqq \delta$, $y \leqq \delta$. This is (6.11).

6.2. Lemma 2. *If* $0<a<b$ *and if, u being fixed, $f(u, v)$ is finite and integrable in v over $(a, b+y)$, then*

$$\int_a^b |\Delta_y f| \frac{dv}{v} \leqq \frac{y}{a} \int_a^{b+y} |d_v f(u, v)|. \tag{6.21}$$

In fact, writing

$$\psi(v) = \int_a^v |d_t f(u, t)|,$$

ψ is measurable and we have, if $\psi(b+y)$ is finite,

$$\int_a^b |\Delta_y f| \frac{dv}{v} \leqq \frac{1}{a} \int_a^b \Delta_y \psi dv \leqq \frac{1}{a} \int_b^{b+y} \psi dv \leqq \frac{y}{a} \int_a^{b+y} |d_v f|.$$

But plainly (6.21) holds if $\psi(b+y)$ is infinite; and this proves the lemma.

6.3. Lemma 3. *If* $0<\delta<\pi$ *and if* (J_T') *holds, then*

$$J_{11} \equiv \int_\delta^{\pi-x} \frac{du}{u} \int_{ky}^{\pi-y} |\Delta_{x,y} f| \frac{dv}{v} = \bar{o}(1).$$

We have, upon applying Lemma 2,

$$J_{11} = \bar{O}\left\{ \int_0^\pi du \int_{ky}^{\pi-y} |\Delta_y f| \frac{dv}{v} \right\}$$

$$= \bar{O}\left\{ \frac{1}{k} \int_0^\pi V(u) du \right\} = \bar{o}(1).$$

6.4. Lemma 4. *If* (J_T') *and* (J_R'') *hold, then*

$$\alpha(x, y; k) = \bar{o}(x).$$

We observe, first, that $f(+0, v)$ exists for nearly all values of v on $(0, \pi)$, for f is of bounded variation as a function of u for almost all values of v on this interval.

We observe, secondly, that $f(+0, v)$ is integrable on $(0, \pi)$. In fact, if u_0 is any number such that $0<u_0<\pi$, we have

$$|f(u, v)| \leqq |f(u_0, v)| + V(v) \text{ for } (u, v) \text{ in } Q.$$

Plainly, then, $f(+0, v)$ is the limit function of a sequence of integrable functions $\{f(u_n, v)\}$, $n=1, 2, \cdots$, satisfying a condition of the type

$$|f(u_n, v)| \leqq V_0(v) \text{ for } 0 \leqq v \leqq \pi,$$

where V_0 is integrable on $(0, \pi)$. The integrability of $f(+0, v)$ now follows from a familiar theorem of Lebesgue.*

We observe, thirdly, that

$$(6.41) \qquad \int_0^x du \int_0^\pi |f(u, v) - f(+0, v)| \, dv = o(x).$$

To prove this we note that

$$B(x, v) \equiv \frac{1}{x} \int_0^x |f(u, v) - f(+0, v)| \, du \leqq W_1(x, v).$$

Thus B tends to zero with x for nearly every v on $(0, \pi)$, and

$$B(x, v) \leqq V(v) \text{ for } 0 \leqq v \leqq \pi.$$

Accordingly, since B is integrable in v for every fixed positive value of x, (6.41) follows from the theorem of Lebesgue mentioned above.†

We observe, finally, that, upon choosing $0<\epsilon$ and $0<\delta<\pi/2$ so that (6.12) holds, we have

$$(6.42) \qquad \int_0^x du \int_{ky}^\delta |\Delta_y f| \frac{dv}{v} \leqq x\epsilon/k \text{ for } x \leqq 2\delta,\ ky \leqq \delta.$$

This results immediately upon applying Lemma 2.

The lemma now follows readily. We have

$$\alpha = \int_0^x du \int_{ky}^\delta |\Delta_y f| \frac{dv}{v} + \bar{O}\left\{ \int_0^x du \int_0^\pi |f(u, v) - f(+0, v)| \, dv \right.$$
$$\left. + x \int_0^{\pi - y} |\Delta_y f(+0, v)| \, dv \right\} = \bar{o}(x)$$

as a consequence of (6.41), (6.42), and the well known fact that

$$\int_0^{\pi - y} |\Delta_y f(+0, v)| \, dv = o(1).$$

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* Lebesgue, 11, p. 375. The full theorem referred to is to the effect that, if $f_n(P)$, $n=1, 2, \cdots$, is integrable on the bounded measurable set E, if

$$|f_n(P)| < \phi(P) \text{ for } n = 1, 2, \cdots, P \text{ on } E,$$

where ϕ is integrable on E, and if

$$\lim_{n \to \infty} f_n(P)$$

exists nearly everywhere in E, then the limit function $f(P)$ of the sequence $\{f_n(P)\}$ is integrable on E, and

$$\lim_{n \to \infty} \int_E f_n(P) dP = \int_E f(P) dP.$$

† It is clear that the conclusion of this theorem remains the same if we replace the discrete variable n by a continuous one.

7.1. **Proof of Theorem III.** We first prove that (J_T) implies (L_R'). For this we choose $0<\epsilon$ and $0<\delta<\pi/2$ so that (6.12) holds; and write

$$\gamma = \left\{ \int_\delta^{\pi-x}\int_{ky}^{\delta} + \int_{kx}^{\delta}\int_\delta^{\pi-y} + \int_\delta^{\pi-x}\int_\delta^{\pi-y} + \int_{kx}^{\delta}\int_{ky}^{\delta} \right\} | \Delta_{x,y}f | \frac{du}{u}\frac{dv}{v}$$
$$= J_{11} + J_{12} + J_{13} + J_{14}, \text{ say}.$$

Now,

$$J_{11} = \bar{o}(1)$$

by (J_T') and Lemma 3, and plainly, by the same reasoning,

$$J_{12} = \bar{o}(1).$$

Further,

$$J_{13} = \bar{o}(1)$$

as is well known. It remains, then, to consider J_{14}.

We write

$$J_{14} = \int_{kx}^{t'\delta}\frac{du}{u}\int_{uy/x}^{\delta} | \Delta_{x,y}f | \frac{dv}{v} + \int_{ky}^{t''\delta}\frac{dv}{v}\int_{vx/y}^{\delta} | \Delta_{x,y}f | \frac{du}{u}$$
$$= J_{14}' + J_{14}'', \text{ say},$$

where

$$t' = \begin{cases} x/y & \text{if } x \leqq y, \\ 1 & \text{if } y < x, \end{cases} \qquad t'' = \begin{cases} 1 & \text{if } x \leqq y, \\ y/x & \text{if } y < x. \end{cases}$$

Then we have

$$J_{14}' = \bar{O}\left[\int_{kx}^{t'\delta}\frac{du}{u}\int_{uy/x}^{\delta}\{ | \Delta_y f(u, v) | + | \Delta_y f(u+x, v) | \}\frac{dv}{v}\right]$$
$$= \bar{O}\left\{x\int_{kx}^{\infty}\frac{du}{u^2}\right\} = \bar{o}(1)$$

upon making use of (6.12) and Lemma 2. In the same way, of course, we get

$$J_{14}'' = \bar{o}(1)$$

and it follows that (J_R) implies (L_R').

The theorem is now immediate. First, since (J_T) implies (C_0) with $s=f(+0, +0)$, (J_T) implies (J_R) with $s=f(+0, +0)$. Next, because (C_1) is common to (J_R) and (L_R), it is plain, by Lemma 4 and what we have just proved, that (J_R) implies (L_R). Thus, since (J_R) implies (1.61) as a consequence of Lemma 1, it follows from Theorem II, part (c), that (J_R) implies

(L_P). Finally, making use of the fact that (J_T) implies (C_0) with $s=f(+0,+0)$ and Theorem II, parts (a) and (b), we find that (J_T) implies (L_1) with $s=f(+0,+0)$. This completes the proof.

8.1. **Lemmas for Theorem IV.** LEMMA 1. *If F satisfies (J_H), then the limits*

$$\operatorname*{limit}_{(u,v)\to(+0,+0)} F(u,v),\quad \lim_{u\to+0}\ \lim_{v\to+0} F(u,v),\quad \lim_{v\to+0}\ \lim_{u\to+0} F(u,v)$$

exist and are equal, and further, if we set

$$P+N=\int_0^u\int_0^v \left| d_{\sigma,t}F(\sigma,t)\right| + \int_0^u \left| d_\sigma F(\sigma,0)\right| + \int_0^v \left| d_t F(0,t)\right| + \left| F(0,0)\right|, \tag{8.11}$$

$$P-N=F \ \textit{for}\ 0<u\leqq\pi,\ 0<v\leqq\pi, \tag{8.12}$$

then both P and N satisfy the following conditions:

$$0\leqq P<A \ \textit{for}\ 0<u\leqq\pi,\ 0<v\leqq\pi, \tag{8.13}$$

$$0\leqq \Delta_x P \quad \textit{for}\ 0<u<u+x\leqq\pi,\ 0<v\leqq\pi, \tag{8.14}$$

$$0\leqq \Delta_y P \quad \textit{for}\ 0<u\leqq\pi,\ 0<v<v+y\leqq\pi, \tag{8.15}$$

$$0\leqq \Delta_{x,y} P \ \textit{for}\ 0<u<u+x\leqq\pi,\ 0<v<v+y\leqq\pi, \tag{8.16}$$

$$P \ \textit{is integrable in}\ Q. \tag{8.17}$$

The proofs of these facts, with the exception of the last, are given by Hardy.* The proof that (8.17) holds can be made to rest on a theorem of Young.† Young proves that, if the conditions (8.13) to (8.16) hold, then P is continuous at every point in the interior of Q, with the possible exception of those points found on a denumerable set of lines, each of which is parallel either to the u- or the v-axis. Thus, assuming Hardy's results, it follows that, if a is any constant, the set of points on which $P<a$ consists of an open set plus, possibly, a set of zero measure. Accordingly, P is measurable in Q; and thus, using (8.13), it follows that (8.17) holds.

8.2. LEMMA 2. *If* (8.14), (8.15), (8.16), *and* (8.17) *hold, and if*

$$0\leqq P<Auv \ \textit{for}\ 0<u\leqq\pi,\ 0<v\leqq\pi, \tag{8.21}$$

then $P/(uv)$ satisfies (L_P') and (L_P'').

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* Hardy, 5, pp. 57–59. Hardy defines P and N in terms of the positive and negative variation of F, but his definitions are equivalent to ours. Hardy states (8.17) without proof.

† Young, 16, p. 31.

We have

$$\zeta = \int_{kx}^{\pi-x} du \int_{ky}^{\pi-y} \left| \Delta_{x,y}\left\{\frac{P}{u^2v^2}\right\}\right| dv$$

$$= \bar{O}\left[\int_{kx}^{\pi-x} du \int_{ky}^{\pi-y} \left\{\frac{\Delta_{x,y}P}{(u+x)^2(v+y)^2} + \frac{y\Delta_x P}{(u+x)^2v^3} + \frac{x\Delta_y P}{u^3(v+y)} + \frac{xyP}{u^3v^3}\right\} dv\right]$$

$$= \bar{O}\left\{xy\int_{kx}^{\pi}\frac{du}{u^3}\int_{ky}^{\pi}\frac{P}{v^3}dv + x\int_{kx}^{\pi}\frac{du}{u^3}\int_{\pi-y}^{\pi}\frac{P}{v^2}dv + y\int_{\pi-x}^{\pi}\frac{du}{u^2}\int_{ky}^{\pi}\frac{P}{v^3}dv\right.$$

$$\left. + \int_{\pi-x}^{\pi}\frac{du}{u^2}\int_{\pi-y}^{\pi}\frac{P}{v^2}dv + \int_{kx}^{(k+1)x}\frac{du}{u^2}\int_{ky}^{(k+1)y}\frac{P}{v^2}dv\right\}$$

$$= \bar{O}\{1/k^2 + y/k + x/k + xy + 1/k^2\} = \bar{o}(1),$$

$$\xi = \bar{O}\left[\int_0^x \frac{du}{u}\int_{ky}^{\pi-y}\left\{\frac{\Delta_y P}{(v+y)^2} + \frac{yP}{v^3}\right\}dv\right]$$

$$= \bar{O}\left\{y\int_0^x\frac{du}{u}\int_{ky}^{\pi}\frac{P}{v^3}dv + \int_0^x\frac{du}{u}\int_{\pi-y}^{\pi}\frac{P}{v^2}dv\right\}$$

$$= \bar{O}(x/k + xy) = \bar{o}(x).$$

Treating η in a similar manner, we conclude the truth of the lemma.

9.1. **Proof of Theorem IV.** It is plain that (Y) implies (Y_P) and (C_1^*). As a consequence of Theorem II, then, it is enough to prove that (Y_P) implies (L_P). For this we note that, if (Y_P) holds, then the function

$$F = uv\phi$$

satisfies (J_H), for, under these circumstances, F is finitely defined everywhere in Q and

$$\int_0^\pi\int_0^\pi |d_{u,v}F| \leqq \int_0^\pi\int_0^\pi |d_{u,v}\{uvf(u,v)\}| + \pi^2|s| < \infty,$$

$$\int_0^\pi |d_\sigma F(\sigma,0)| = 0, \int_0^u\int_0^v |d_t F(0,t)| = 0.$$

We note, further, that, on defining P and N as in Lemma 1, we have

$$P + N = \int_0^\pi |d_{u,v}(uvf)| + uv|s| < Auv \text{ for } 0 < u \leqq \pi,\ 0 < v \leqq \pi.$$

Applying Lemmas 1 and 2 now, we see that ϕ satisfies (L_P') and (L_P''). Since (C_1) is common to (L_P) and (Y_P), the proof is complete.

10.1. **Proof of Theorem V.** We have, if $1 < p_1$, $1 < p_2$,

$$\gamma = \bar{O}\left[\left\{\int_0^{\pi-x} du \int_0^{\pi-y} |\Delta_{x,y}f|^{p_1} dv\right\}^{1/p_1}\left\{\int_{kx}^{\infty}\frac{du}{u^{q_1}}\int_{ky}^{\infty}\frac{dv}{v^{q_1}}\right\}^{1/q_1}\right] \qquad \left(\frac{1}{p_1}+\frac{1}{q_1}=1\right)$$

$$= \bar{O}\{x^{1/p_1}y^{1/p_1}(kx)^{1/q_1-1}(ky)^{1/q_1-1}\} = \bar{O}(k^{-2/p_1}) = \bar{o}(1),$$

$$\alpha = \bar{O}\left[\left\{\int_0^{x} du \int_0^{\pi-y} |\Delta_{y}f|^{p_2} dv\right\}^{1/p_2}\left\{\int_0^{x} du\int_{ky}^{\infty}\frac{dv}{v^{q_2}}\right\}^{1/q_2}\right] \qquad \left(\frac{1}{p_2}+\frac{1}{q_2}=1\right)$$

$$= \bar{O}\{x^{1/p_2}y^{1/p_2}x^{1/q_2}(ky)^{1/q_2-1}\} = \bar{O}(xk^{-1/p_2}) = \bar{o}(x),$$

and

$$\gamma = \bar{O}\left\{\frac{1}{k^2xy}\int_0^{\pi-x}\int_0^{\pi-y} |\Delta_{x,y}f|\, dv\right\} = \bar{O}(1/k^2) = \bar{o}(1),$$

$$\alpha = \bar{O}\left\{\frac{1}{ky}\int_0^{x} du\int_0^{\pi-y} |\Delta_{y}f|\, dv\right\} = \bar{O}(x/k) = \bar{o}(x)$$

if $p_1=p_2=1$. Treating β in a similar manner and noting that (C_1) is common to (L_R) and (HL), we deduce the truth of the theorem.

11.1. **Proof of Theorem VI.** Since (J_H) implies (C_0) with $s=f(+0,+0)$ and f is finitely defined everywhere in Q, we have only to prove that f satisfies (1.51), (HL'), and (HL''), the last two with $p_1=p_2=p_3=1$.

Consider (1.51). We have

$$\begin{aligned}|\Delta_{x,y}(uvf)| &\leqq (u+x)(v+y)|\Delta_{x,y}f| + (u+x)y|\Delta_x f| \\ &\quad + x(v+y)|\Delta_y f| + xy|f| \\ &\leqq ab|\Delta_{x,y}f| + ay\left\{|\Delta_x f(u,0)| + \int_u^{u+x} d\sigma\int_0^b |d_{\sigma,t}f(\sigma,t)|\right\} \\ &\quad + xb\left\{|\Delta_y f(0,v)| + \int_0^a\int_v^{v+y}|d_{\sigma,t}f|\right\} \\ &\quad + xy \underset{0\leqq\sigma\leqq a,0\leqq t\leqq b}{\text{maximum}} |f(\sigma,t)|\end{aligned}$$

for $0\leqq u<u+x\leqq a$, $0\leqq v<v+y\leqq b$. Thus

$$\int_0^x\int_0^y |d_{u,v}(uvf)| \leqq xy\left\{3\int_0^\pi\int_0^\pi |d_{u,v}f| + \int_0^\pi |d_u f(u,0)| + \int_0^\pi |d_v f(0,v)| + \underset{0\leqq u\leqq\pi,0\leqq v\leqq\pi}{\text{maximum}} |f|\right\} < Axy.$$

This is (1.51).

Turning now to (HL') and (HL''), we define P and N as in Lemma 1, §8.1. Then, using the results of that lemma, we have

$$\int_0^{\pi-x} du \int_0^{\pi-y} |\Delta_{x,y}P|\, dv = O\left\{ \int_0^{\pi-x} du \int_0^{\pi-y} \Delta_{x,y}P dv \right\}$$
$$= O\left\{ \int_0^{x} du \int_0^{y} P dv + \int_{\pi-x}^{\pi} du \int_{\pi-y}^{\pi} P dv \right\}$$
$$= O(xy),$$
$$\int_0^{x} du \int_0^{\pi-y} |\Delta_y P|\, dv = O\left\{ \int_0^{x} du \int_{\pi-y}^{\pi} P dv \right\} = O(xy).$$

Treating N and the other integral in (HL'') in the same manner, we infer the truth of the theorem.

12.1. **Lemmas for Theorem VII.** LEMMA 1. *If F satisfies (J_H) and is absolutely continuous* in $(a, a; \pi, \pi)$ for every $0 < a < \pi$, then F coincides in the region $0 < u \leqq \pi$, $0 < v \leqq \pi$ with a function G which is absolutely continuous in Q.*

We first define P and N as in Lemma 1, §8.1, for $0 < u \leqq \pi$, $0 < v \leqq \pi$, and note that $P(u, +0)$, $P(+0, v)$, and $P(+0, +0)$ exist, the first for every u and the second for every v, on $(0, \pi)$. We complete the definition of P by setting

$$P(u, 0) = P(u, +0) \text{ for } 0 < u \leqq \pi,\ P(0, v) = P(+0, v) \text{ for } 0 < v \leqq \pi,$$
$$P(0, 0) = P(+0, +0).$$

We next note that

$$(12.11)\qquad \begin{aligned} &\int_0^{x} |d_u P(u, \pi)| = o(1), && \int_0^{x}\int_0^{\pi} |a_{u,v}P| = o(1),\\ &\int_0^{y} |d_v P(\pi, v)| = o(1), && \int_0^{\pi}\int_0^{y} |d_{u,v}P| = o(1). \end{aligned}$$

In fact, upon making use of (8.14) and (8.15), our definition of P on the axes, and the appropriate limiting processes, we find that

* By definition F is absolutely continuous in $(a, a; \pi, \pi)$ if (i) the functions $F(u, 0)$ and $F(0, v)$ are absolutely continuous on (a, π), and if (ii), corresponding to each $0 < \epsilon$, we can so choose $0 < \delta$ that, if $\{(x_i', y_i'; x_i'', y_i'')\}$, $i = 1, 2, \cdots$, is any collection of rectangles contained in $(a, a; \pi, \pi)$, no two of which have a common interior point, and the total measure of which is less than δ, then

$$\sum_{i=1}^{\infty} |f(x_i'', y_i'') - f(x_i'', y_i') - f(x_i', y_i'') + f(x_i', y_i')| < \epsilon.$$

This definition is equivalent to Carathéodory's, 2, p. 653, but is different from Hobson's, 8, p. 346, which requires only that (b) be satisfied.

$$0 \leqq \Delta_x P(u, \pi) \text{ for } 0 \leqq u < u + x \leqq \pi,$$
$$0 \leqq \Delta_{x,y} P \qquad \text{for } 0 \leqq u < u + x \leqq \pi, 0 \leqq v < v + y \leqq \pi.$$

Thus we have

$$\int_0^x \left| d_u P(u, \pi) \right| = P(x, \pi) - P(0, \pi) = o(1),$$

$$\int_0^x \int_0^\pi \left| d_{u,v} P \right| = P(x, \pi) - P(0, \pi) - P(x, 0) + P(0, 0)$$
$$= o(1) - P(x, 0) + P(0, 0) = o(1)$$

since

$$P(0, 0) = P(+0, +0) = \lim_{x \to +0} \lim_{y \to +0} P(x, y) = \lim_{x \to +0} P(x, 0).$$

This proves that the first two relations in (12.11) hold. The last two can, of course, be proved correct in a similar manner.

To complete the proof of the lemma, we now define N on the axes, as we did P, in terms of its limiting values, and we set

$$G = P - N \text{ for } (u, v) \text{ in } Q.$$

Then plainly N satisfies (12.11), and therefore so also does G. But

(12.12) $$G = F \text{ for } 0 < u \leqq \pi, 0 < v \leqq \pi,$$

and therefore G is absolutely continuous in $(a, a; \pi, \pi)$ for every $0 < a < \pi$. We conclude from these two properties of G that G is absolutely continuous in Q. By (12.12), this proves the lemma.

12.2. LEMMA 2. *If g is integrable over Q, then uvg satisfies (L_2).*

We have

$$\zeta_1 = \int_x^{\pi - x} du \int_y^{\pi - y} \left| \Delta_{x,y} g \right| dv = o(1),$$

$$\xi_1 = \int_0^x u du \int_y^{\pi - y} \left| \Delta_y g \right| dv = O\left\{ x \int_0^\pi \int_0^{\pi - y} \left| \Delta_y g \right| dv \right\} = o(x).$$

Similarly,

$$\eta_1 = o(y).$$

12.3. LEMMA 3. *If $h(u)$ is integrable over $(0, \pi)$, and if $g(u, v)$ is integrable over Q, then the function $uH(u, v)$, where*

$$H(u, v) = h(u) - \int_v^\pi g(u, t)dt,$$

satisfies (L_2).

We have

$$\zeta_1 = \int_x^{\pi-x} du \int_y^{\pi-y} \left| \Delta_{x,y}\left\{\frac{H}{v}\right\} \right| dv$$

$$= O\left\{ y\int_x^{\pi-x} |\Delta_x h|\, du \int_y^{\pi-y} \frac{dv}{v^2} + y\int_x^{\pi-x} du \int_y^{\pi-y} \frac{dv}{v^2}\int_v^\pi |\Delta_x g(u, t)|\, dt \right.$$

$$\left. + \int_x^{\pi-x} du \int_y^{\pi-y} \frac{dv}{v}\int_v^{v+y} |\Delta_x g(u, t)|\, dt \right\}$$

$$= O\left\{ \int_0^{\pi-x} |\Delta_x h|\, du + \int_0^\pi du \int_0^{\pi-y} |\Delta_x g(u, t)|\, dt \right\} = o(1).$$

Further,

$$\xi_1 = O\left\{ y\int_0^x u\,du \int_y^{\pi-y} \frac{dv}{v^2}\int_v^\pi |g(u, t)|\, dt + \int_0^x u\,du \int_y^{\pi-y} \frac{dv}{v}\int_v^{v+y} |g(u, t)|\, dt \right\}$$

$$= O\left\{ x\int_0^x du \int_0^\pi |g(u, t)|\, dt \right\} = o(x).$$

Finally,

$$\eta_1 = O\left\{ \int_0^y dv \int_x^{\pi-x} |\Delta_x h|\, du + \int_0^y dv \int_x^{\pi-x} du \int_v^\pi |\Delta_x g(u, t)|\, dt \right\}$$

$$= O\left\{ y\int_0^{\pi-x} |\Delta_x h|\, du + y\int_x^{\pi-x} du \int_0^\pi |\Delta_x g(u, t)|\, dt \right\} = o(y).$$

This completes the proof.

13.1. **Proof of Theorem VII.** By our hypothesis, F can be so defined on the axes as to satisfy (J_H). Moreover, F obviously is absolutely continuous in $(a, a; \pi, \pi)$ for every $0 < a < \pi$. Thus, by Lemma 1, F coincides in the region $0 < u \leqq \pi$, $0 < v \leqq \pi$ with a function G absolutely continuous in Q.

Now, since G is absolutely continuous in Q, there exist functions $g(u, v)$, $h(u)$, $l(v)$, the first integrable over Q and the last two over $(0, \pi)$, such that, for (u, v) in Q, G is given by*

* See Hobson, 8, pp. 592, 615, or Carathéodory, 2, p. 654.

$$G(u, v) = \int_u^\pi d\sigma \int_v^\pi g(\sigma, t)dt - \int_u^\pi h(\sigma)d\sigma - \int_v^\pi l(t)dt + G(\pi, \pi)$$
$$= g_1 - h_1 - l_1 + G(\pi, \pi), \text{ say.}$$

We proceed to express f in terms of g, h, and l.

We have, for $0 < u < u + x \leqq \pi$, $0 < v < v + y \leqq \pi$,

$$\Delta_{x,y}\left\{ \int_0^u d\sigma \int_0^v f dt \right\} = \Delta_{x,y}(uvG)$$
$$= (u + x)(v + y)\Delta_{x,y} g_1 + (u + x)y\Delta_x G + x(v + y)\Delta_y G + xyF.$$

Hence, upon dividing each member of this equation by xy, setting $y = x$, and letting x tend to zero, we have*

$$f(u, v) = uvg + uH + vL + F, \tag{13.11}$$

where

$$H(u, v) = h(u) - \int_v^\pi g(u, t)dt,$$

$$L(u, v) = l(v) - \int_u^\pi g(\sigma, v)d\sigma,$$

for almost all (u, v) in Q.

The theorem now follows. Since g and h are integrable, uvg and uH satisfy (L_2) by Lemmas 2 and 3. Similarly, since l is integrable, vL satisfies (L_2). Applying Theorem II, then, we see that (L_1) holds with f replaced by $uvg + uH + vL$ and s, by zero. On the other hand, since F satisfies (J_H), it satisfies (Y) with $s = F(+0, +0)$ by Theorem VI. Hence, by Theorem IV, F satisfies (L_1) with $s = F(+0, +0)$. Combining these results with (13.11), we reach the desired conclusion.

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